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The equivalence of two face-centred icosahedral tilings with respect to local derivability

Johannes Roth†

Institut für Theoretische und Angewandte Physik, Universität Stuttgart, Pfaffenwaldring 57, D-7000 Stuttgart 80, Federal Republic of Germany

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Abstract. We demonstrate how two tilings with face-centred icosahedral symmetry can be derived from one another by using only local information. The tilings under consideration are the zonohedral tiling proposed by Socolar and Steinhardt, which is closely related to the original three-dimensional rhombohedral tiling, and the tetrahedral tiling of Danzer. Both tilings have matching rules, and our proof is based on this property.

1. Introduction

Since Penrose described an aperiodic pattern with fivefold symmetry in 1974, a large number of tilings has been discovered and various generation methods for different non-crystallographic symmetries have been described. Some patterns look very different but contain the same information. Penrose's original dart and kite tiling, for example, is fully equivalent to rhombus tiling as well as to pentagon tiling: each one can be derived from the other.

In this paper we do not want to add a further tiling to the zoo of patterns, but demonstrate the equivalence of two quite different tilings, the zonohedral tiling of Socolar and Steinhardt [1] and the tetrahedral tiling of Danzer [2, 3] using the concept of mutual local derivability formulated by Baake *et al* [4]. Both tilings are examples not of the usual simple icosahedral quasilattices but of face-centred icosahedral (FCI) ones‡. Interest in FCI quasicrystals has increased because of the discovery of stable AlFeCu FCI quasicrystals. Recently the structure of these quasicrystals has been investigated with neutron diffraction by Cornier-Quiquandon *et al* [5]. They showed that the atomic arrangement may be described by a tiling model.

Kramer *et al* [6, 7] systematically derived new tilings for icosahedral quasicrystals from the face-centred hypercubic lattice, often called D_6 in root lattice terminology. They projected either the three-dimensional boundaries of the Delauney cells or of the Voronoi cells of this lattice to get the tilings. They proved that it is possible to derive locally Danzer's tetrahedral tiling from the Voronoi cell projection of the D_6 tiling [9]. Here, on the other hand, we demonstrate the equivalence of Danzer's tiling and the tiling of Socolar and Steinhardt.

† Present address: LASSP, Clark Hall, Cornell University, Ithaca, NY 14853, USA.

‡ The simple and FCI quasicrystals have different inflation/deflation properties. The factor for the first one is τ^3 whereas it is τ for the second. τ is the golden number $(\sqrt{5}+1)/2$.

The paper is organized as follows. After a short description of the properties of both tilings relevant for the present discussion we describe how to obtain the tetrahedral tiling, starting from the zonohedra, and then we discuss the opposite case. Finally we mention some applications of the results.

Before starting we shall mention briefly what mutual local derivability means: given two quasiperiodic tilings it is possible to derive one from the other by only considering finite parts of the first tiling and describing rules to generate the second one from the first. The same must be true if one starts from the second tiling and derives the first. For a precise definition of mutual local derivability and its relation to the local isomorphism the reader is advised to consult [4].

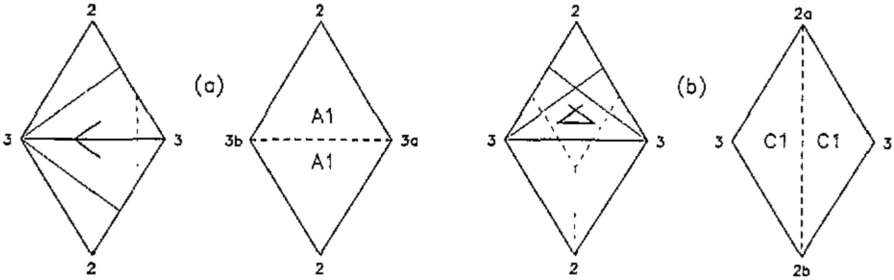
2. Zonohedral tiling

A zonohedron as defined by Coxeter [8] is the convex hull of the sum of all vectors of a vector star. Its faces are built up by $2m$-gons, if the vector star contains groups of m coplanar vectors. In our case $m = 2$ and therefore all faces are rhombi. The zonohedral tiling of Socolar and Steinhardt [1] was derived in a paper about the generalized dual method (GDM) as the Penrose local isomorphism (PLI) class of icosahedral quasicrystals. The GDM is very similar to the original grid method but it uses planes with two spacings, τ and 1, and the sequence is generated by the Fibonacci series. The PLI class is defined by the existence of simple local matching rules, simple inflation and deflation rules, and by a simple decoration of the tiles which forms a linear quasiperiodic grid called the Ammann quasilattice. The Ammann quasilattice for the PLI class is extremely singular, which means that more than three grid planes meet in a single intersection point. Contrary to the two-dimensional case the singularities cannot be eliminated, they can almost be shifted. The duals of non-singular intersection points are the well known Ammann rhombohedra, but only the prolate rhombohedra (PR) occur in the PLI class. Normally, the singularities are resolved by arbitrary infinitesimal shifts of the grid planes and then the grid is dualized to generate the ordinary rhombohedra. Socolar and Steinhardt decided, however, not to eliminate the singularities, which would have destroyed the PLI properties, but to dualize the singular intersection points by zonohedra which are the hull of all possible resolutions of the singular points. For fourfold, fivefold, and sixfold degeneracy they obtained the rhombic dodecahedron (RD), the rhombic icosahedron (RI) and the rhombic triacontahedron (RT). Three-dimensional pictures of the polyhedra may be found in Socolar and Steinhardt [1].

The rhombic dodecahedron has the same volume as two PR and two oblate rhombohedra (OR) and D_{2h} symmetry. The RI can be divided into five PR and five OR and has D_{5d} symmetry, and the RTC may be filled with ten PR and ten OR and displays the full icosahedral Y_h symmetry.

It is possible to construct local clusters of zonohedra that have complete icosahedral symmetry. The first cluster is the RT itself, the second a star of twenty PR. The three complete packings with a (single) centre of icosahedral point symmetry are generated by inflation of the clusters. One of the packings has an RT at its centre, the next shell being composed of thirty RD. The other two have a star of PR rhombohedra at their centres, one having twelve RI as the next shell, the other having twelve RT.

The matching rules can be obtained by decorating the tiling with Ammann planes, which are generated by inflation and rescaling of the quasiperiodic gridplanes. There are three different possible intersections of the Ammann planes with the rhombic faces of the zonohedra: the cuts form an arrow (figure 1(a)), a triangle (figure 1(b)) or the planes intersect in a single line (figure 1(c)). Socolar's and Steinhardt's symbols are smaller versions of the cutting figure. To build up the tiling, equivalent symbols must match. The matching rules applied to the zonohedra partially break the symmetry: the RD has only C_{2h} symmetry and the RI has C_{5h} symmetry (see also section 4).



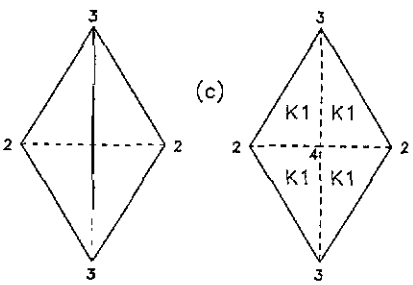


Figure 1. The three different rhombic faces: the picture at the left-hand side of each part (a)–(c) displays the intersection of the faces with the Ammann planes (thin lines) and the symbol used by Socolar and Steinhardt (thick lines). The broken lines denote the subdivision into pieces. The picture at the right-hand side shows the corresponding tetrahedra faces of Danzer's tiling. Broken lines are the edges of the tetrahedra. The numbers and letters are taken from [2, 3]. The letters give the type of tetrahedra and the number behind the letter the vertex class of the fourth corner of the tetrahedron which does not lie in the plane.

Socolar and Steinhardt also describe the deflation of the zonohedral tiling. The deflation takes place in two steps: the original tiles are at first divided into *pieces*, then the pieces are rejoined to form again zonohedra scaled down by a factor of $1/\tau$. The general procedure is quite complicated because the same pieces may serve different roles in the new zonohedra. The deflation procedure is nevertheless deterministic, the local deflation being determined by the Ammann planes and the adjoining cells.

Socolar and Steinhardt did not realize that their tiling is *not* a simple icosahedral tiling but a face-centred one. This may easily be proven either by the fact that the inflation–deflation ratio is τ (compared with τ^3 in the simple icosahedral case) or by splitting the vertices into even and odd ones with respect to the sum of their six-dimensional coordinates and calculating their relative occurrence.

3. Tetrahedral tiling

The tetrahedral tiling of Danzer [2, 3] consists of four tetrahedra called A, B, C and K, out of the set of all tetrahedra that can be constructed by cuts of the mirror planes of the full icosahedral point symmetry group. The tiling has the following properties.

(i) The edges of the tiles are coloured red, green or white, if their dihedral angles are 90°, 60° or 120°, or multiples of 36°.

(ii) The vertices fall into four classes indicated by numbers. The class number 4 only occurs at the corners of the K-tiles where three red edges meet. There is only one possible arrangement of tiles in class 4.

(iii) If the red-coloured edges are removed from the tiling, four tetrahedra of type A, B and C and eight tetrahedra of type K are combined to form octahedra of type A, B, C and K, respectively.

The matching rules for the faces of the tiling are purely geometric, if none of the edges of a face is red. But if there is a red edge, then the tiles on both sides of the face must be mirror images. This allows the tiling to be described by octahedra.

There are exactly three tilings which are globally symmetric with respect to the full icosahedral group. By inflation and subsequent expansion they are permuted cyclically.

Figures of all the faces of the tetrahedra may be found in Danzer's [3] paper together with a description of the construction of the whole tetrahedra.

4. From zonohedra to tetrahedra

In this section we shall describe how the zonohedra must be subdivided to yield the tetrahedra and how the matching rules are translated from the zonohedral tiling to the tetrahedral tiling.

We first consider the decoration of the zonohedral faces and how they must be decorated by tetrahedra (figure 1) to obey Danzer's matching rules. The rhombi with the arrow and the triangle seem to have a symmetry lower than the A and C tetrahedra decorating them, but taking into account the deflation rules for the A1 and C1 faces given by Danzer, the higher symmetry of the tetrahedron decoration is broken, and the uniqueness is restored. Symmetry breaking may be indicated by assigning letters a and b to the corners of the tetrahedra with the same number. The rhombus with the line (line rhombus) is decorated by four K-tetrahedra. The vertices of the rhombi all belong to classes 2 and 3 and the centre of a line rhombus is a vertex of class 4. The decoration of the faces with tetrahedra obeys Danzer's matching rules in the following way: the tiles on both sides of a rhombus are mirror images, since each rhombus contains at least one red coloured tetrahedron edge and the tetrahedra adjoined to a rhombus form an octahedron. We therefore have a one-to-one correspondence of the arrow, triangle and line symbol to the A, C and K octahedra or tetrahedra, which is indeed a translation of Socolar and Steinhardt's into Danzer's matching rules.

We next consider the subdivision of the zonohedra. The decoration of the line rhombi with K-tetrahedra fixes all vertices in the interior of the PR and RD. The RT has an additional vertex at its centre, and the RI a vertex lying at the centre of the semi-spherical cap built up by line rhombi. No other vertices occur. All the vertices in the interior of the zonohedra are of type 1. If all the vertices of the tetrahedra and the decoration of the faces of the zonohedra are known, the edges of the tetrahedra can be drawn immediately and the derivation is complete. The subdivision of the zonohedra into tetrahedra is shown in figures 2(a)–(d), where most of the tetrahedra can be seen directly. The RD is the only zonohedron which has additional A- and C-tetrahedra not shown in the cuts, but these tetrahedra can be deduced from the faces. The subdivision of the zonohedra into tetrahedra is also summarized in table

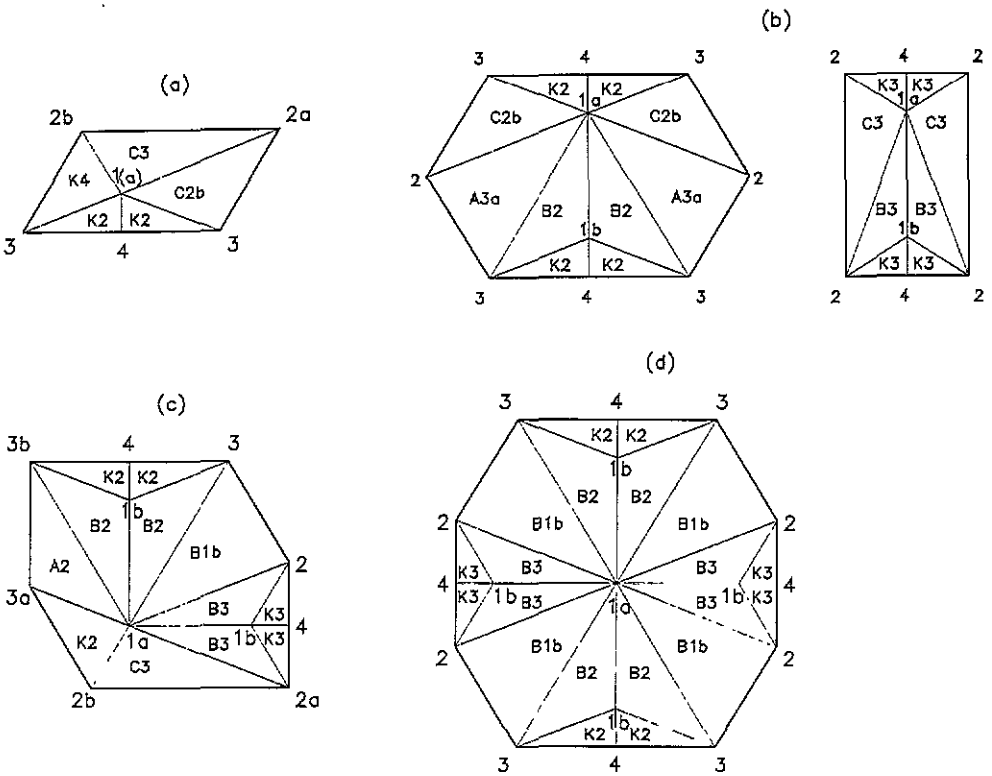


Figure 2. The zonohedra and their decomposition into tetrahedra: (a) prolate rhombohedron, (b) rhombic dodecahedron, (c) rhombic icosahedron, (d) rhombic triacontahedron. The pictures display two-dimensional representative cuts along mirror planes of the three-dimensional polyhedra. In the case of the dodecahedron we give two perpendicular cuts containing the vertical 4–1–1–4 axes. The planes through the polyhedra show all the tetrahedra of a certain zonohedron except for the dodecahedron. The uniqueness of the subdivision may be checked by considering the cut planes in connection with the decoration of the faces given in figure 1. The marking for the faces is the same as in figure 1, the lower case letters distinguish the vertices if a tetrahedron has two corners of the same type.

Table 1. The subdivision of Socolar's and Steinhardt's zonohedra (SSZ) into Danzer's tetrahedra (DT). The indices are the number of tetrahedra in an asymmetric part of the zonohedron.

DT \ SSZ	PR	RD	RI	RT
A	0	8_2	10_1	0
B	0	4_1	40_4	120_1
C	6_1	12_3	10_1	0
K	12_2	8_2	40_4	120_1

1, where the number of tetrahedra in the complete zonohedron and in an asymmetric unit are given.

We have demonstrated how the zonohedral tiling can be subdivided into Danzer's tetrahedra. The matching rules for the tetrahedra inside the zonohedra are fulfilled by the rules for subdivision and the matching rules on the surface have been translated into rules for a unique decoration of the rhombic faces of the zonohedra with tetrahedra which obey the matching rules for the tetrahedral tiling. Therefore any of Socolar's and Steinhardt's zonohedral tiling can be transformed into a tetrahedral tiling.

5. From tetrahedra to zonohedra

Now we shall describe the derivation of the zonohedral tiling from a tetrahedral tiling. The tetrahedral tiling should be a global tiling as defined by Danzer [3], which means that it is a covering of the space without overlap.

The first step of our derivation is to change the tetrahedral tiling into an octahedra tiling by removing the red coloured edges, which is always possible as proved by Danzer [2, 3]. The vertices of class 4 are eliminated in this way. Danzer also showed that the tiling consisting of A, B, C and K tiles can be transformed into a A, C, K and τK tiling, where τK is the union of a B and a K tile. In this way the vertices of class 1b of the B-octahedra are eliminated (see figure 2). All the remaining vertices of class 1 will now be called class 1a vertices. Each octahedron of type A, C, K and τK has at least two mirror planes, and one of these planes contains the vertices numbers 2 and 3. An inspection of the octahedra shows that the vertices are the corners of rhombi of a unique size. Thus we have found the faces of the zonohedra. The vertices of class 1 lie symmetrically above and below the mirror plane spanned by the class 2 and 3 vertices.

If we now take an arbitrary 1a vertex, we find that it is connected only to class 2 and 3 vertices, since each vertex of an octahedron is linked only with vertices of another class. The rhombi spanned by the class 2 and 3 vertices must generate a closed polyhedral shell since the octahedra are packed face-to-face.

The last step would be to show that the cells are indeed zonohedra, but this can be checked by inspection of the possible environments of class 1a vertices. There are exactly four different environments which correspond exactly to the four zonohedra. A calculation of the frequencies of the 1a environments and the zonohedra shows that the numbers for both tilings coincide.

The zonohedral tiling is nothing else but a Voronoi decomposition of the quasilattice of class 1a vertices of Danzer's tiling, since the class 2 and 3 vertices lie in mirror planes between 1a vertices. This fact shows that the derivation is definitely local.

We have described how the tetrahedral tiling must be modified to a octahedral tiling which can immediately be transformed into a zonohedral tiling.

6. Discussion and conclusions

We have demonstrated the equivalence of two FCI tilings with respect to mutual local derivability: any tiling of Socolar and Steinhardt's zonohedra obeying their matching rules can be transformed into Danzer's tetrahedral tiling obeying his matching rules and vice versa.

The equivalence of quasiperiodic patterns is often very useful for the construction of a tiling. If one is interested for example in the frequencies of certain patterns or in vertex configurations it is sometimes helpful to know an equivalent pattern (with a different construction rule) and to do the calculations there. The derivation of the vertex configurations and tile frequencies for example is very complicated in the zonohedral tiling but easily obtained in the tetrahedral tiling. Therefore it seems to be a necessary task to classify the known quasicrystalline patterns with respect to mutual local derivability, as stated by Baake *et al* [4]; the result will be the essential quasicrystalline tiling patterns and a classification of the large 'zoo' of tilings.

Acknowledgment

The author wishes to thank M Baake for helpful comments.

Note added in proof. After submission of this letter I was informed about a preprint by Professor Danzer, in which similar claims are made [10].

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